Journal of Knot Theory and Its Ramifications
Vol. 19, No. 6 (2010) 799–807

DOI: 10.1142/S0218216510008133

THE EINSTEIN–PODOLSKY–ROSEN PROPOSITION ON QUANTUM THEORY AND ITS IMPLICATIONS FOR INTUITIVE CLASSICAL LOGIC*

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Accepted 18 May 2010

ABSTRACT

Einstein, Podolsky and Rosen raised foundational questions about the completeness of quantum mechanics, if certain intuitive logical statements regarding the nature of reality were assumed to be true. These questions are ultimately of significance to the information content of the theory, which is currently the focus of intense research. In this presentation, selected investigations that have made progress in addressing the EPR concerns and that shed light on the nature of quantum states are surveyed. The implications for intuitive classical logic are speculated in the concluding remarks.

Keywords: EPR; quantum theory; intuitive logic.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

The short but seminal paper of Einstein–Podolsky–Rosen (EPR) in 1935 has been the source of much speculation, but also motivation for much research on the perplexing features of quantum theory that seem to be at odds with our classically oriented intuition about nature [1]. In their 1935 paper, Einstein, Podolsky and Rosen exposed inconsistencies between the predictions of quantum mechanics and a reasonable rational perception of reality, which opened up new directions in the investigation of the foundations of quantum theory. The efforts continue to this day; in fact, the attempts to resolve the inconsistencies proposed by EPR have led to a sharper understanding of both quantum theory and our understanding of the nature of reality in the intervening years. Here, we provide a brief survey of both the EPR results and the salient features of a few other investigations that challenge some of

*This paper is based on a longer invited talk, *The Einstein–Podolsky–Rosen Paradox and Quantum Mysteries a la Mermin*, presented at the Conference on Knot Theory and Its Ramifications — Quantum Knots in Washington XXVIII (February 25–March 1, 2009).

the EPR assumptions. This presentation does not make any original contribution to the field other than some speculative remarks in the concluding section. Section 2 will provide a brief review of the EPR paper, and Sec. 3 will do the same for the works of Bohm–Aharanov, Bell, Greenberger–Horne–Zeilinger, Hardy and Mermin. Also, this presentation will not provide a comprehensive review of the major works since the EPR publication. Rather, this discussion is selective and pedagogical in nature. Space limitations and the limited scope of this work would not allow for considering some important results, albeit in some cases controversial, obtained by Von Neumann, Bohm, DeWitt, Everett, Aspect, Wheeler, Zurek and others.

2. EPR Proposition in Brief

In setting the stage for the basic premises of their proposition on quantum theory, Einstein, Podolsky and Rosen state that *correctness* and *completeness* should be the main criteria in judging a successful physical theory [1]. According to EPR: "The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement". It is mainly the *completeness* criterion for quantum mechanics that EPR address in their paper. It is stipulated that a necessary requirement for a complete theory is that "every element of physical reality must have a counterpart in the physical theory. The elements of the physical reality cannot be determined by a priori philosophical considerations, but must be found by an appeal to results of experiments and measurements". Furthermore, EPR settle for a"reasonable" (not comprehensive) definition of reality: "*If, without in any way disturbing a system, we can predict with certainty* (*i.e. with probability equal to unity*) *the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity*". EPR concede that as a sufficient condition, this criterion agrees with both classical and quantum mechanical notions of reality. They elucidate the case for quantum mechanics through the description of a single particle motion with the concept of a quantum mechanical *state*, and how such a state, "completely characterized by a wave function", provides information on the momentum and position of the particle. Following the formalism of quantum mechanics EPR reach the usual conclusion, encapsulated by the *uncertainty principle*, that "when the momentum of a particle is known, its coordinate has no physical reality", the generalization of which leads to similar statements on non-commuting operators. "... if the operators corresponding to two physical quantities ... do not commute ... then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first".

This leads EPR to a crucial proposition in the form of the following statements: "*Either* (1) *the description of reality given by the wave function in quantum mechanics is not complete; or* (2) *when the operators corresponding to two physical* *quantities do not commute the two quantities cannot have simultaneous reality*. For if both of them had simultaneous reality — and thus definite values — these values would enter into the complete description, according to the condition of completeness". Assuming, as most believe to be true, that the wave function of a state does encompass the physical reality of the system under question, EPR then proceed to demonstrate that the assumption of completeness of quantum mechanics and imposition of the criterion of reality lead to a contradiction by considering in some detail the behavior of a two-component system under the laws of quantum theory.

We capture the essence of the EPR formulation following closely the derivation in the EPR paper: Consider two quantum mechanical systems (I and II) that interact for a time period T, but then the interaction is turned off for all $t > T$. To know the state of each system after the interaction has been turned off, according to quantum theory, further measurements must be performed. Let $\Psi(1,2)$ represent the state of the system in terms of the complete set of eigenstates, $u_i(1)$ and $\psi_i(2)$, of its two subsystems.

$$\Psi(1,2) = \sum_{i=1}^{\infty} \psi_i(2) u_i(1). \tag{2.1}$$

Now, a measurement on system I will lead to a reduction of the wave packet, leading to the determination of one of the eignevalues, say, a_r for the eigenstate $u_r(1)$, which means only one term from the previous sum will survive, also forcing the state of system II into a particular eigenstate, in this case, $\psi_r(2)$.

$$\Psi(1,2) = \psi_r(2) u_r(1). \tag{2.2}$$

Had we chosen to measure a different physical quantity for System I, using $v_i(1)$ as the corresponding eigenstates for that quantity, an expansion of the total wave function of the system similar to (2.1) with $\phi_i(2)$ as coefficients of $v_i(1)$ and following the same procedure, would have yielded a different state for system II.

$$\Psi(1,2) = \phi_r(2) v_r(1). \tag{2.3}$$

EPR then proceeded to show that it was possible for the two different eigenstates of system II to correspond to two non-commuting conjugate variables (Q and P). According to EPR, this indicates that it is possible to assign two different wave functions to the same reality, i.e. system II after interaction with system I had ceased. And, the two wave functions could represent non-commuting variables. Under the initial EPR proposition above, either (1) quantum theory is not complete, or (2) when the operators for two physical quantities do not commute, the two cannot have simultaneous reality. But, the reverse was demonstrated for (2) in the case of the two systems (I and II) hypothesized by EPR, which lead the authors to the conclusion that (1) must hold, i.e. quantum theory is not complete.

Having concluded the incompleteness of quantum theory in terms of a complete description of physical reality, and leaving open the question of whether such a theory exists, EPR strike a hopeful note with the belief that "such a theory is possible [1]".

3. Brief Survey of Selected Work Since EPR

In this section, we provide brief descriptions of investigations that emerged due to the EPR proposition in somewhat historical (chronological) order.

3.1. *Bohm–Aharanov realization of an EPR experiment*

The interest in pursuing an experimental realization of the EPR system was instrumental to the work of Bohm and Aharanov in 1957 [2]. Bohm and Aharonov suggested a realization of the two systems proposed by EPR by considering a molecule of total spin zero consisting of two oppositely aligned atoms of spin 1/2. Once the molecule disintegrates and far from the source region, measurement of a component of the spin of one atom would determine the spin state of the other atom instantaneously. While this is possible classically (all spin components commute), in quantum mechanics it poses several problems including random fluctuations of the other spin components of the second atom (remembering that spin operators are represented by Pauli matrices that do not commute). Furthermore, the particular configuration of this experiment introduces a potential for a violation of causality that is imposed through Einstein's formulation of the special theory of relativity, disallowing the possibility of instantaneous interactions (effects) for space-like events.

3.2. *Bell's inequalities*

The EPR proposition (or "paradox" as preferred by some authors) implied that quantum mechanics could not be a complete theory, leading some investigators to suspect that seemingly random fluctuations associated with the quantum phenomena might be due to variables that were not explicitly present in the theory — implying that additional variables (hitherto hidden) might restore to the theory some of the desirable and deterministic features of classical theories and that concepts such as causality and locality that inform our classical intuition of natural phenomena could be restored.

In studying the EPR proposition, evidently, it is the stipulation that a measurement on one system should not have an effect on another system with which it no longer interacts that poses the main difficulty, i.e. loosely stated, it is the requirement of locality and implicit separability of non-interacting systems that poses the most serious conceptual problem. Although an explicit hidden variable interpretation of quantum mechanics has been constructed (Bohm), it contains a strong non-local structure [3, 4]. Bell was able to demonstrate that any such theory that reproduces the predictions of quantum mechanics has the characteristic of *non-locality* [5, 6].

Bell constructed a theory of quantum mechanics using a single or a set of hidden variables satisfying the requirements of EPR for local reality, using a system of two spin 1/2 particles (Bohm–Aharonov pairs). Bell then proceeded to derive an inequality for statistical correlations based on expectation values of the product of measurement outcomes (joint probabilities of the outcomes), with $E(\overrightarrow{a}, \overrightarrow{b})$ designating the expectation value of joint outcomes along the directions $\overrightarrow{a}\, and\, \overrightarrow{b}$:

$$|E(\overrightarrow{a}, \overrightarrow{b}) - E(\overrightarrow{a}, \overrightarrow{c})| - E(\overrightarrow{b}, \overrightarrow{c}) - 1 \leq 0. \tag{3.1}$$

There are indeed a set of inequalities that can be derived for various situations. Bell then showed that there are choices of directions a, b, and c for which quantum mechanics does violate the derived inequality. In fact if the directional vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are in the x-y plane with the azimuthal angles zero, 60 and 120 degrees, respectively, one would obtain:

$$|E(\overrightarrow{a}, \overrightarrow{b}) - E(\overrightarrow{a}, \overrightarrow{c})| - E(\overrightarrow{b}, \overrightarrow{c}) - 1 = \frac{1}{2} > 0. \tag{3.2}$$

Note that in a system of two spin 1/2 particles, contradictions develop (violations of Bell's inequalities) only when quantum mechanical statistical predictions are considered (and not in all situations).

3.3. *The Greenberger–Horne–Zeilinger multi-particle approach*

By contrast, Greenberger–Horne–Zeilinger (GHZ) using three entangled particles showed that quantum mechanical predictions for certain measurements are in conflict with local realism in cases where quantum theory makes definite predictions, i.e. non-statistical disagreements with local realism are found [7, 8]. Therefore, the quantum mechanical predictions of three-photon GHZ states are in stronger conflict with the requirement of local realism than the two-particle state analysis of Bell.

We follow closely the arguments presented in the references [9, 10]. Consider a maximally entangled three photon state $|\Psi\rangle$, with polarization states designated by H (horizontal) and V (vertical), and with each photon (identified by subscripts 1, 2, or 3 on the kets in the equation below) in a superposed state.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_2|H\rangle_3 + |V\rangle_1|V\rangle_2|V\rangle_3). \tag{3.3}$$

Consider specific predictions following from this state for polarization measurements on each photon in two other bases; one rotated relative to the original by 45 degrees (H' and V'), and one in a circular polarization base (L and R: left-handed and right-handed). The H/V, H'/V', and R/L bases are eigenfunctions of the Pauli operators σ_z , σ_x, and σ_y in the z, x, and y directions, respectively, with eigenvalues +1 or -1. The new H'/V' and R/L set may be obtained from the original H/V set:

$$|H'\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \quad \text{and} \quad |V'\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \tag{3.4}$$

and with

$$|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle), \quad \text{and} \quad |L\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle). \tag{3.5}$$

Let us designate the joint outcome of the measurement of the y-component of the spin of one particle, the y-component of the spin of another particle, and the x-component of the spin of a third particle by "yyx". Suppose a "yyx" measurement is performed on the three photons, with circular polarization measurements on photons 1 and 2 (i.e. "yy") and linear polarization (45 degrees relative to the original) measurement on photon 3 (i.e. "x"). The initial three-photon state (3.3) will then transform, with the help of (3.4) and (3.5), to the following:

$$|\Psi\rangle \to \frac{1}{2}(|R\rangle_1|L\rangle_2|H'\rangle_3 + |L\rangle_1|R\rangle_2|H'\rangle_3 + |R\rangle_1|R\rangle_2|V'\rangle_3 + |L\rangle_1|L\rangle_2|V'\rangle_3). \tag{3.6}$$

Note that in (3.6) any specific individual or two-photon joint measurement is maximally random. For example, photon "1" will exhibit polarization states R or L with the same probability of 50%. This is true of photon "2", and in polarization states of H' or V' for photon "3". Also, note that every term in the sum yields a product of (-1) for eigenvalues in a "yyx" measurement. Therefore, it is possible to know the polarization of a 3rd photon exactly, if measurements are made on the other two. Hence, if both photons 1 and 2 are found to be in a right-handed circular state, then photon 3 must be in a V' state.

These predictions are independent of spatial separations of the photons and the relative time-order of the measurements. Therefore, one could also perform simultaneous measurements on the three photons in a conveniently chosen reference frame, where the particles are far apart from each other, obtaining the same results. Symmetry of the system dictates that by cyclic permutation analogous expressions are obtained in case of two other experiments ("xyy" and "yxy") with the role of the photons interchanged:

$$|\Psi\rangle \to \frac{1}{2}(|R\rangle_3|L\rangle_2|H'\rangle_1 + |L\rangle_3|R\rangle_2|H'\rangle_1 + |R\rangle_3|R\rangle_2|V'\rangle_1 + |L\rangle_3|L\rangle_2|V'\rangle_1), \tag{3.7}$$

$$|\Psi\rangle \to \frac{1}{2}(|R\rangle_1|L\rangle_3|H'\rangle_2 + |L\rangle_1|R\rangle_3|H'\rangle_2 + |R\rangle_1|R\rangle_3|V'\rangle_2 + |L\rangle_1|L\rangle_3|V'\rangle_2). \tag{3.8}$$

To explain the perfect correlations predicted by the wave function (3.6) and its permutations for the previous experiments (3.7) and (3.8), given the requirement of locality (stipulated by EPR), one is led to assume that each photon carries elements of reality for both "x" and "y" measurements that determine the outcome of the results, otherwise, it would be necessary to concede that information could be conveyed with a speed faster than the speed of light in violation of the special theory of relativity [9]. For a given photon "i", let X_i with values $+1(-1)$ designate the element of reality for H'(V') polarization, and Y_i with values $+1(-1)$ designate the element of reality for R(L) polarization.

In order to reproduce the results of the previous GHZ experiments, one concludes that $X_1Y_2Y_3 = -1$, $Y_1X_2Y_3 = -1$, and $Y_1Y_2X_3 = -1$. Suppose another experiment is performed measuring linear H'/V' polarization on all three photons (i.e. an "xxx" measurement). Given the above notion of elements of local reality carried by each photon, and the EPR notion that any specific measurement for "x" must be independent of whether an "x" or "y" measurement is performed on the other photons, one should obtain $X_1X_2X_3 = -1$. This is inferred by noting that $Y_iY_i = +1$ for all "i", therefore, using this fact, one may rewrite: $X_1X_2X_3 = (X_1Y_2Y_3)(Y_1X_2Y_3)(Y_1Y_2X_3)$, which results from the judicious insertion of Y_iY_i , and proper grouping of the terms. Using the known results of the previous experiments for the value of the products inside each parenthesis, one obtains $X_1X_2X_3 = (X_1Y_2Y_3)(Y_1X_2Y_3)(Y_1Y_2X_3) = (-1)(-1)(-1) = -1$. This means that imposition of the criterion of local reality allows for prediction of specific terms that must be present in the wave function of the three-photon state of GHZ-type "xxx" experiments. If arguments of local realism are used to generate the quantum state of an "xxx" measurement by considering all possible allowable photon states, based on the results and observations of maximally correlated pairs in previously carried out experiments, the following terms must be included:

$$|\Psi\rangle \to |V'\rangle_1|V'\rangle_2|V'\rangle_3 + |H'\rangle_1|H'\rangle_2|V'\rangle_3 + |H'\rangle_1|V'\rangle_2|H'\rangle_3 + |V'\rangle_1|H'\rangle_2|H'\rangle_3. \tag{3.9}$$

However, if we transform the original state given in (3.3) in terms of new linear polarization basis states of H'/V' (for an "xxx" measurement) using the usual quantum mechanical operations, we would obtain:

$$|\Psi\rangle \to |H'\rangle_1|H'\rangle_2|H'\rangle_3 + |H'\rangle_1|V'\rangle_2|V'\rangle_3 + |V'\rangle_1|H'\rangle_2|V'\rangle_3 + |V'\rangle_1|V'\rangle_2|H'\rangle_3. \tag{3.10}$$

A quick comparison of the terms in (3.9) and (3.10) reveals that none of the terms generated through our intuition of local reality is the same as the terms in the actual quantum mechanical expression! There is maximal discrepancy between quantum mechanics and intuitive classical notions of locality in this case. In fact, quantum theory would always predict $X_1X_2X_3 = +1$, whereas classical intuition would always give $X_1X_2X_3 = -1$. The quantum mechanical GHZ predications are in fact reflected in experimental results [10].

3.4. *Mermin's pedagogical approach*

The above GHZ three-particle analysis was recast by Mermin in the form of a Gedanken experiment to reveal a stark difference between the findings of intuitive local realism and that of quantum theory [11]. Later, Mermin developed a similar Gedanken experiment based on the work of Hardy for two-particle states that considerably simplifies the GHZ version, and shows a sharp conflict with the conjectures of local realism [12, 13]. Mermin's Gedanken demonstration is based on the analysis of the findings of black boxes that act as detectors, but in an arrangement that excludes any connection between the boxes or between the boxes and the source. Unfortunately, space limitations will not allow a detailed presentation of Mermin's and Hardy's works that have a significant pedagogical value. More details can be found in Ref. [14].

4. Concluding Remarks

The EPR definition for the *completeness* of a physical theory appears to rest on a sense of *reality* that has its roots in the intuitive classical notions of *locality* and *separability*. We have observed both through Bell's theorem (inequalities) and through Bell's theorem without inequalities (GHZ results) that at the level of phenomena behaving in accordance with the laws of quantum theory, our intuitive classical understanding of nature fails, whereas the quantum mechanical predictions have so far been verified by experimental results. The rational propositions of EPR, based on an intuitive logic that codifies experiences through an understanding of nature in accordance with classical physical theories — i.e. intuitive classical logic — appear to fall short in comparison to the predictions of quantum theory. This fact suggests that a new intuition based on different logical elements is required for a better and more accurate understanding of natural phenomena. Through such an intuition, it should be possible to explain the classical phenomena in terms of quantum mechanical concepts, but one should not expect the opposite.

Foundational questions regarding quantum mechanics still remain. For example, in the EPR analysis, the role of the reduction of the wave packet was taken for granted. Yet, it is not fully understood even today. The Copenhagen interpretation (reduction of the wave packet) leads to statements regarding the information content of quantum systems subject to any measurement process. This has consequences for the amount, availability, and accessibility of information in nature. The ideas of entanglement are playing an increasingly important role in the elucidation of the measurement process. Interest in the practical aspects of quantum information and quantum computations have placed special focus on the resolution of difficult foundational problems, requiring all the tools available in the arsenal of physics, mathematics, computer science, and related fields. Quantum knots, braids, and many other interesting concepts might be needed, and may, indeed, be essential in tackling some of the more intractable foundational problems.

Other foundational questions of interest to mathematics are worth considering. For example, why does the square modulus of quantum amplitudes represent the probabilistic behavior of natural phenomena, and what other equivalent measure could be envisioned or constructed to connect the theory with experiment? Is the information content of a quantum system a conserved quantity, albeit not easily accessible? The resolution of these and many other problems would require a sharper understanding of the behavior of quantum systems interfacing and interacting with other quantum systems of various degrees of complexity.

Acknowledgment

I am indebted to Professor William C. Parke for many discussions on the various topics presented here, the shortcomings and errors are, of course, all mine.

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